GROTHENDIECK CONFERENCE
IHES, JAN 2009
MALTSINIOTIS: Grothendieck and homotopical algebra

Writings of Grothendieck:
- Pursuing stacks (1983)
- Les derivateurs (1990)

Homotopical themes of Grothendieck:
A) ∞-stacks
B) Modelisators ("modélisateurs")
C) Abelianisation and schematization
D) Homotopical constructions in Cat
E) Derivators
F) Model categories

$$(\infty\text{-groupoids}) \underset{\Pi_\infty}{\overset{|\cdot|}{\rightleftarrows}} (\mathrm{Top})$$

<u>Conjecture</u> (Grothendieck)

$$W^{-1}_{\infty\text{-eq}}(\infty\text{-groupoids}) \underset{\sim}{\overset{\sim}{\rightleftarrows}} W^{-1}_{\mathrm{Top}}(\mathrm{Top}) = \mathrm{Hot}$$

$\simeq$ CW complexes / homotopy

(Not proved)

A modelisator is a pair (M, W) s.t. $W^{-1}M \simeq Hot$

Classical examples:

(Top, W_{Top}) $\quad$ $(SSets, W_{simpl.})$ $\quad$ (Cat, W_{Cat})

Here: Δ = category of simplizes
$SSets = \hat{\Delta} = \Delta^{op}Sets$ = category of simplicial sets = category of presheaves on Δ

W_{simpl} = { morphisms of simplicial sets for which the topological realization is a topological weak equivalence }

Cat = category of small categories

W_{Cat} = { functors for which the nerve is a simplicial weak equivalence }

(Reference to Illusie's thesis)

<u>Def</u> (Grothendieck)

$Hot := W_{Cat}^{-1} Cat$

where $W_{Cat} := \{ u : A \to B \mid (u^*, u_*) : \hat{A} \to \hat{B}$ is an Artin-Mazur equivalence $\}$

A morphism of toposes $f = (f^*, f_*) : X \to Y$ is an Artin-Mazur equivalence if for every locally constant sheaf L on Y, we have

$$H^i(Y, L) \xrightarrow{\sim} H^i(X, f^* L)$$

where

L = sheaf of sets for $i = 0$
sheaf of groupes for $i = 1$
sheaf of abelian groups for $i \geq 2$

Let X be a topos. A morphism $F \to G$ of sheaves is said to be a weak equivalence if $X/_F \longrightarrow X/_G$ is an Artin-Mazur equivalence

<u>Special case</u>: $X = \hat{A}$

$F \to G$ morphism of presheaves on A

$F \to G$ weak equivalence $\underset{\text{def}}{\iff}$

$$\begin{array}{ccc} \hat{A}/F & \longrightarrow & \hat{A}/G \\ \wr\wr & & \wr\wr \\ \widehat{A/F} & \longrightarrow & \widehat{A/G} \end{array}$$

is an Artin-Mazur equivalence

$$i_A : \hat{A} \longrightarrow Cat$$
$$F \longmapsto A/F$$
$$W_A := i_A^{-1}(W_{cat})$$

If $A = \Delta$ then $W_\Delta = W_{simpl}$

<u>Question</u>: Under what hypotheses is $(\hat{A}, W_A)$ a modelisator, i.e. $W_A^{-1}\hat{A} \simeq Hot$
or more precisely: when is $W_A^{-1}\hat{A} \to W_{Cat}^{-1} Cat = Hot$
(the functor induced by $i_A : F \mapsto A/F$)
an equivalence of categories?

$i_A : \hat{A} \to Cat$ admits a right adjoint

$$j_A : Cat \longrightarrow \hat{A}$$
$$C \longmapsto (a \mapsto Hom_{Cat}(A/a, C))$$

<u>Definition</u> (Grothendieck)
A small category A is a weak test category if $j_A(W_{Cat}) \subset W_A$ and $W_A^{-1}\hat{A} \underset{\bar{j}_A}{\overset{\bar{i}_A}{\rightleftarrows}} W_{Cat}^{-1} Cat$
(induced by i_A and j_A) are equivalences of categories, quasi-inverse to each other.

<u>Terminology</u>
A small category C is aspherical if $(C \to e = *) \in W_{Cat}$
A functor $A \to B$ is aspherical if $\forall b \in Ob(B)$,
A/b is an aspherical category
A presheaf F on A is aspherical if $i_A F := A/F$ is an aspherical category
A morphism $F \to G$ of presheaves is aspherical if $i_A F \to i_A G$ is an aspherical functor.

Proposition (Grothendieck)
Let A be a small category. TFAE:
i) A is a weak test category
ii) A is aspherical and $j_A(W_{cat}) \subset W_A$
iii) For all small categories C, the adjunction morphism $i_A j_A(C) \longrightarrow C$ is aspherical
iv) For all small categories C, the adjunction morphism $i_A j_A(C) \longrightarrow C$ is in W_{cat}
v) For all small categories C admitting a final object, the presheaf $j_A(C)$ is aspherical

Definition (Grothendieck)
- A is a local test category if for all $a \in Ob(A)$, A/a is a weak test category
- A is a test category if it is a local test category and a weak test category

Examples of test categories
- The category Δ (simplicial)
- The category $\square$ (cubical)
- The Θ category of Joyal

Example of a weak test cat. which is <u>not</u> a test cat:
The subcategory Δ^f of Δ generated by the face operators.

Terminology

A presheaf F on A is locally aspherical
$\underset{\text{def}}{\Longleftrightarrow}$ $\forall a \in Ob(A)$, the presheaf $F|A/a$ on A/a is aspherical
$\Longleftrightarrow$ $\forall a \in Ob(A)$, the presheaf $a \times F$ on A is aspherical

A segment in $\hat{A}$ is a presheaf I on A equipped with two sections

$$e_{\hat{A}} \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} I$$

over the final object $e_{\hat{A}}$ of $\hat{A}$.
It is separating if $Ker(\delta_0, \delta_1) = \emptyset$

The Lawvere object of a topos is the subobject classifier.
The Lawvere object of $\hat{A}$ is defined by

$$L_{\hat{A}} = j_A(\{0 \to 1\})$$

The empty and full subobjects of $e_{\hat{A}}$ define sections

$$e_{\hat{A}} \rightrightarrows L_{\hat{A}}$$

which makes $L_{\hat{A}}$ a separating segment, the Lawvere segment.

<u>Theorem</u> (Grothendieck)
Let A be a small category. TFAE:
i) A is a local test category
ii) $L_{\hat{A}}$ is locally aspherical
iii) There exists a locally aspherical separating segment in $\hat{A}$

Moreover, if these conditions are satisfied, then A is a test category if and only if A is aspherical

<u>Corollary 1</u>
Suppose A is a small category with finite products, and suppose I is an object of A with two sections

$$e \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} I$$

over the final object e, such that there is no commutative square in A of the form:

$$\begin{tikzcd} a \arrow[r] \arrow[d] & e \arrow[d, "\delta_1"] \\ e \arrow[r, "\delta_0"] & I \end{tikzcd}$$

Then A is a test category.

<u>Corollary 2</u>

The product of a local test category with an arbitrary category is a local test category. The product of a test category with an aspherical category is a test category.

<u>Theorem</u> (Grothendieck - Cisinski)

Let A be a small category. TFAE:

i) A is a local test category

ii) $\hat{A}$ admits a closed model structure, in which W_A is the class of weak equivalences, and monomorphisms are the cofibrations.

Moreover, if these conditions are satisfied, there is an equivalence of categories induced by i_A

$$\mathrm{Ho}\,\hat{A} := W_A^{-1}\hat{A} \xrightarrow{\sim} (W_{Cat}/A)^{-1}(Cat/A)$$

Derivators

Let $\mathcal{A}$ be an abelian category, and $\mathrm{Der}(\mathcal{A})$ its derived category. For any small category I, $\underline{\mathrm{Hom}}(I^{op}, \mathcal{A})$ (the functor category) is also an abelian category, and we can consider

$$\mathbb{D}(I) := \mathrm{Der}(\underline{\mathrm{Hom}}(I^{op}, \mathcal{A}))$$

Any functor $f : I \to J$ gives rise to $f^* : \mathbb{D}(J) \to \mathbb{D}(I)$. The assignment $I \mapsto \mathbb{D}(I)$ defines a contravariant 2-functor from small categories to categories.

Note: $\mathrm{Der}(\mathcal{A}) = \mathbb{D}(e)$

Definition: A derivator is a 2-functor

$$I \mapsto \mathbb{D}(I), \qquad I \to J \quad \mapsto \quad \mathbb{D}(J) \xrightarrow{f^*} \mathbb{D}(I)$$

$$\begin{tikzcd} I \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & {} \end{tikzcd} \longmapsto \begin{tikzcd} \mathbb{D}(I) \arrow[r, phantom, "\Uparrow\alpha^*"] & \mathbb{D}(J) \arrow[l, bend right, "f^*"'] \arrow[l, bend left, "g^*"] \end{tikzcd}$$

satisfying certain axioms, the most important being:

- existence of $f_! : \mathbb{D}(I) \to \mathbb{D}(J)$ left adjoint to f^*
- existence of $f_* : \mathbb{D}(I) \to \mathbb{D}(J)$ right adjoint to f^*

Let $\mathcal{C}$ be a category,

$W \subset \mathrm{Mor}(\mathcal{C})$ "weak equivalences"

I small category

$W_I \subset \mathrm{Mor}(\underline{\mathrm{Hom}}(I^{op}, \mathcal{C}))$ weak equivalences "objectwise"

The 2-functor $I \mapsto W_I^{-1}\underline{\mathrm{Hom}}(I^{op}, \mathcal{C}) =: \mathbb{D}(I)$ is viewed as encoding the nonadditive "triangulated structure" on $\mathrm{Ho}(\mathcal{C}) := \mathbb{D}(e) = W^{-1}\mathcal{C}$

Alex Heller (1988) : homotopical theory

$\left.\begin{array}{l}\text{Bernhard Keller (1991)}\\ \text{Jens Franke (1996)}\end{array}\right\}$: additive variants

Triangulated derivators (2001)

K-theory of triangulated derivators (2002)

Conjectures: Comparison, Additivity, Localisation

$\left.\begin{array}{l}\text{Cisinski}\\ \text{Neeman}\end{array}\right\}$ Additivity (2005)

Muro (2007) : Comparison for K_1

$\mathcal{C} = \mathcal{C}at$

$W = W_{\mathcal{C}at}$

I small category

$HOT(I) := \mathbb{D}(I) := W_I^{-1} \underline{Hom}(I^{op}, \mathcal{C}at)$

This defines a derivator HOT

$HOT(e) = \mathcal{H}ot$

<u>Notation</u>

$I \xrightarrow{f} J \quad \longmapsto \quad HOT(J) \xrightarrow{f^*} HOT(I)$

$f_!, f_* : HOT(I) \longrightarrow HOT(J)$

<u>Definition</u> (Grothendieck)

$f : I \to J$ functor between small categories
Say that f is smooth (resp. proper) if
for every cartesian square

$$\begin{tikzcd} I' \arrow[r, "g"] \arrow[d, "f'"'] & I \arrow[d, "f"] \\ J' \arrow[r, "h"] & J \end{tikzcd}$$

the base change morphism

$$f^* h_* \longrightarrow g_* f'^* \qquad (\Leftrightarrow f'_! g^* \longrightarrow h^* f_!)$$

$$(\text{resp. } h^* f_* \to f'_* g^* \qquad (\Leftrightarrow g_! f'^* \longrightarrow f^* h_!))$$

is an isomorphism, and this property still is true after any base change.

<u>Theorem</u> (Grothendieck)

Let $f : I \to J$ be a functor. TFAE:

i) f is smooth

ii) For every diagram of cartesian squares

$$\begin{tikzcd} I'' \arrow[r, "g"] \arrow[d] & I' \arrow[r] \arrow[d] & I \arrow[d, "f"] \\ J'' \arrow[r, "h"'] & J' \arrow[r] & J \end{tikzcd}$$

if h is aspherical then so is g

iii) For every $j \in Ob(J)$ the canonical functor

$$I_j \to j \backslash I, \quad i \mapsto (i, j \xrightarrow{1_j} f(i) = j)$$

is aspherical.

<u>Corollary</u>

$f : I \to J$ is smooth $\iff$ $f^{op} : I^{op} \to J^{op}$ is proper

END OF LECTURE.